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Almost periodic solutions of impulsive Hopfield neural networks with periodic delays

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Summary. An impulsive Hopfield neural network with delay which differs from a constant by a small amplitude periodic perturbation is considered. If the corresponding system with constant delay has an isolated $\omega-$periodic solution and the period of the delay is rationally independent with ω, then under suitable assumptions it is proved that in a sufficiently small neighbourhood of this orbit the perturbed system has a unique almost periodic solution.

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1 INTRODUCTION

A neural network is a network that performs computational tasks such as associative memory, pattern recognition, optimization, model identification, signal processing, etc. on a given pattern via interaction between a number of interconnected units characterized by simple functions. From the mathematical point of view, an artificial neural network corresponds to a nonlinear transformation of some inputs into certain outputs. Many types of neural networks have been proposed and studied in the literature and the Hopfield-type network has become an important one due to its potential for applications in various fields of daily life. The model proposed by Hopfield, also known as Hopfield's graded response neural network, is based on an analogue circuit consisting of capacitors, resistors and amplifiers.

Hopfield neural networks have found applications in a broad range of disciplines [1, 2, 3] and have been studied both in and discrete time cases by many researchers. Most neural networks can be classified as either continuous or discrete. In spite of this broad classification, there are many real world systems and natural processes that behave in a piecewise continuous style interlaced with instantaneous and abrupt changes (impulses). Periodic dynamics of the Hopfield neural networks is one of the realistic and attractive modellings for the researchers. Signal transmission between the neurons causes time delays. Therefore the dynamics of Hopfield neural networks with discrete or distributed delays have a fundamental concern.

In the present paper we consider an impulsive Hopfield neural network with delay which differs from a constant by a small amplitude periodic perturbation [5]. If the corresponding system with constant delay has an isolated $\omega-$periodic solution and the period of the delay is rationally independent with ω, then under suitable assumptions it is proved that in a sufficiently small neighbourhood of this orbit the perturbed system has a unique almost periodic solution.

2 STATEMENT OF THE PROBLEM. MAIN RESULTS

We consider a Hopfield neural network with impulses at fixed instants and delay fluctuating around a constant value which may be assumed 1 without loss of generality:

$$\begin{gathered}\dot{x}_i(t) = -a_i(t)x_i(t)+\sum_{j=1}^{m} b_{ij}(t)f_j(x_j(t))+\sum_{j=1}^{m} g_{ij}(t,x_j(t-1-\varepsilon\varphi(t)))+J_i(t),\ t\neq t_k,\\ \Delta x_i(t_k) = I_{ik}(x(t_k),x(t_k-1-\varepsilon\varphi(t))),\quad k\in\mathbb{Z}, \qquad (1)\\ \Delta x_i(t)=0 \ \text{ if } \ t-1-\varepsilon\varphi(t)=t_k \ \text{ for some } \ k\in\mathbb{Z},\ t\notin\{t_k\}_{k\in\mathbb{Z}},\ i=\overline{1,m},\end{gathered}$$

where m is the number of neurons in the network, $x=(x_1,x_2,\ldots,x_m)^T\in\Omega\subset\mathbb{R}^m$, $x_i(t)$ is the state of the i-th neuron at time t, $a_i(t)>0$ is the rate at which the i-th neuron resets its state when isolated from the system, $b_{ij}(t)$ is the synaptic connection weight from the j-th neuron to the i-th one, $f_j(\cdot)$ are signal transmission functions of the j-th neuron, the terms $g_{ij}(t,\cdot)$ reflect the synaptic connection from the j-th neuron to the i-th one with transmission delay $1+\varepsilon\varphi(t)$, $\varepsilon\in[0,\varepsilon_0)$ is a small parameter, ε_0 will be specified below, $J_i(t)$ is the external input to the i-th neuron, $\mathbb{Z}$ is the set of all integers, $\Delta x_i(t_k)\equiv x_i(t_k+0)-x_i(t_k-0)$ are the impulses at instants t_k and $\{t_k\}_{k\in\mathbb{Z}}$ is a strictly increasing sequence such that $\lim_{k\to\pm\infty} t_k=\pm\infty$.

As usual in the theory of the impulsive differential equations [4], at the points of discontinuity t_k of the solution $x(t)=(x_1(t),\ldots,x_m(t))^T$ we assume that $x(t_k)\equiv x(t_k-0)$. It is clear that, in general, the derivatives $\dot{x}(t_k)$, $k\in\mathbb{Z}$, do not exist. On the other hand, there exist the limits $\dot{x}(t_k\pm0)$. According to the above convention, we assume $\dot{x}(t_k)\equiv\dot{x}(t_k-0)$.

Similarly, the derivative $\dot{x}$ does not exist at the other points of discontinuity of the right-hand side of the differential equation in (1), *i.e.*, at points t which are solutions of the equations

$$t-1-\varepsilon\varphi(t)=t_k, \qquad (2)$$

$k \in \mathbb{Z}$. We require the continuity of the solution $x(t)$ at such points if they are distinct from the instants of impulse effect t_k.

In the sequel we require the fulfillment of the following assumptions:

A1. The functions $a_i(t) > 0$, $b_{ij}(t)$, $J_i(t)$ $(i, j = \overline{1,m})$ are continuous and $\omega-$periodic.

A2. The functions $f_j(\cdot)$ are continuously differentiable in Ω_j- the projection of the domain Ω on the j-th axis, with locally Lipschitz continuous first derivatives.

A3. The functions $g_{ij}(t, y_j)$ $(i, j = \overline{1,m})$ are continuous and $\omega-$periodic with respect to t, continuously differentiable with respect to y_j, with locally Lipschitz continuous with respect to y_j first derivatives.

A4. The functions $I_{ik}(x, y)$ $(i \in \overline{1,m},\ k \in \mathbb{Z})$ are continuously differentiable with respect to $x, y \in \Omega$, with locally Lipschitz continuous with respect to x, y first derivatives.

A5. There exists a positive integer p such that $t_{k+p} = t_k + \omega$, $I_{i,k+p}(x, y) = I_{ik}(x, y)$ for $k \in \mathbb{Z},\ i = \overline{1,m}$ and $x, y \in \Omega$.

A6. The function $\varphi(t)$ is ω_1-periodic, where ω_1/ω is irrational, and Lipschitz continuous:

$$|\varphi(t') - \varphi(t'')| \le K|t' - t''|, \quad t', t'' \in \mathbb{R}.$$

If $\varepsilon_0 \le \min\ \{1,\ 1/K\}$, then for $\varepsilon \in (0, \varepsilon_0)$ equation (2) has a unique solution $t_k(\varepsilon)$ for each $k \in \mathbb{Z}$. It obviously satisfies

$$|t_k(\varepsilon) - t_k - 1| \le \varepsilon, \quad t_k(0) = t_k + 1.$$

It is natural to assume that the period ω is distinct from the unperturbed delay 1. For the sake of definiteness we assume that $\omega > 1$ and $t_i \neq 0\ \forall k \in \mathbb{Z}$.

For $\varepsilon = 0$, from (1) we obtain

$$\begin{aligned}\dot{x}_i(t) = -a_i(t)x_i(t) + \sum_{j=1}^{m} b_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^{m} g_{ij}(t, x_j(t-1)) + J_i(t), \quad t \neq t_k,\\ \Delta x_i(t_k) = I_{ik}(x(t_k), x(t_k - 1)), \quad k \in \mathbb{Z},\\ \Delta x_i(t_k + 1) = 0 \ \text{ if } \ t_k + 1 \neq t_\ell\ \forall \ell \in \mathbb{Z},\ i = \overline{1,m},\end{aligned} \tag{3}$$

so called *generating system*, and suppose that

A7. The generating system (3) has an $\omega-$periodic solution $\psi(t)$ such that $\psi(t) \in \Omega$ for all $t \in \mathbb{R}$.

A8. $\frac{\partial g_{ij}}{\partial y_j}(t, y_j)\Big|_{y_j = \psi_j(t-1)} = 0$ for $t \in \mathbb{R}$, $\frac{\partial}{\partial y_j} I_{ik}(\psi(t_k), y)\Big|_{y = \psi(t_k - 1)} = 0$ for $k \in \mathbb{Z}$, $i, j = \overline{1,m}$.

Now define the linearized system with respect to $\psi(t)$:

$$\begin{aligned}\dot{x}(t) &= A(t)x(t), \qquad t \neq t_k,\\ \Delta x(t_k) &= B_k x(t_k), \qquad k \in \mathbb{Z},\end{aligned} \tag{4}$$

where

$$A(t) = -\text{diag}\ (a_i(t))_{i=1}^{m} + \left(b_{ij}(t)\frac{df_j}{dx_j}(\psi_j(t))\right)_{i,j=1}^{m},$$

$$B_k = \left(\frac{\partial}{\partial x_j} I_{ik}(x, \psi(t_k - 1))\Big|_{x = \psi(t_k)}\right)_{i,j=1}^{m}.$$

Let $X(t)$ be the fundamental matrix of system (4) (see [4]). Denote

$$\Lambda = \frac{1}{\omega}\ \ln\ X(\omega), \qquad \Phi(t) = X(t)e^{-\Lambda t}.$$

$\Phi(t)$ is an $\omega-$periodic piecewise continuous nondegenerate matrix-valued function, with points of discontinuity of the first kind at $\{t_k\}_{k\in\mathbb{Z}}$. Now we make two additional assumptions:

A9. The matrices $E+B_k$, $k\in\mathbb{Z}$, are nonsingular (E – the unit matrix).
A10. The matrix Λ has no eigenvalues with real part zero.

Together with (4) we consider the nonhomogeneous system

$$\begin{aligned} \dot{z}(t) &= A(t)z(t)+h(t), \qquad t\neq t_k,\\ \Delta z(t_k) &= B_k x(t_k)+c_k, \qquad k\in\mathbb{Z}, \end{aligned} \tag{5}$$

where $h(t)\in AP_m\{t_k\}$ – the space of all almost periodic functions with values in $\mathbb{R}^m$, which are piecewise continuous with points of discontinuity of the first kind at $t_k,\ k\in\mathbb{Z}$, while $c_k\in ap_m$ – the space of all almost periodic sequences with values in $\mathbb{R}^m$ [4]. Under these assumptions system (5) has a unique almost periodic solution (see [4, Theorem 25.3]).

We give only those fragments of the proof that will be used henceforth.

Let n be the number of eigenvalues of Λ with negative real parts. Without loss of generality we may assume that $\Lambda=\operatorname{diag}(P,N)$, where P and N are square matrices of order $m-n$ and n, respectively, such that

$$\operatorname{Re}\lambda_j(P)>0,\ j=\overline{1,m-n},\quad \operatorname{Re}\lambda_j(N)<0,\ j=\overline{m-n+1,m}.$$

Denote

$$G(t)=\begin{cases} -\operatorname{diag}(e^{Pt},0) & \text{for } t<0,\\ \ \ \operatorname{diag}(0,e^{Nt}) & \text{for } t>0. \end{cases}$$

It can be shown that

$$\|G(t)\|\le Ce^{-\alpha|t|},$$

where C and α are positive constants. Moreover,

$$z(t)=\int_{-\infty}^{+\infty}\Phi(t)G(t-\tau)\Phi^{-1}(\tau)h(\tau)\,d\tau+\sum_{k\in\mathbb{Z}}\Phi(t)G(t-t_k)\Phi^{-1}(t_k)c_k \tag{6}$$

is the unique almost periodic solution of (5). We shall also need the estimates

$$\int_{-\infty}^{+\infty}\|G(t-\tau)\|\,d\tau\le\frac{2C}{\alpha}\quad\text{and}\quad\sum_{k\in\mathbb{Z}}\|G(t-t_k)\|\le\frac{2Cp}{1-e^{-\alpha\omega}}.$$

The first one follows from

$$\int_{-\infty}^{\infty}e^{-\alpha|t-\tau|}\,d\tau=\int_{-\infty}^{\infty}e^{-\alpha|\sigma|}\,d\sigma=2\int_0^{\infty}e^{-\alpha\sigma}\,d\sigma=\frac{2}{\alpha}.$$

To derive the second one, without loss of generality we may assume that $t_0\le t<t_1$. Then

$$\sum_{k\in\mathbb{Z}}\|G(t-t_k)\|\le C\sum_{k=0}^{\infty}e^{-\alpha(t-t_{-k})}+C\sum_{k=1}^{\infty}e^{-\alpha(t_k-t)}.$$

For the first sum we have

$$\sum_{k=0}^{\infty}e^{-\alpha(t-t_{-k})}=e^{-\alpha t}\sum_{k=0}^{\infty}e^{\alpha t_{-k}}=e^{-\alpha t}\sum_{\lambda=0}^{\infty}\sum_{\nu=0}^{p-1}e^{\alpha t_{-\nu-\lambda p}}=e^{-\alpha t}\sum_{\lambda=0}^{\infty}\sum_{\nu=0}^{p-1}e^{\alpha(t_{-\nu}-\lambda\omega)}$$

$$=e^{-\alpha t}\sum_{\nu=0}^{p-1}e^{\alpha t_{-\nu}}\sum_{\lambda=0}^{\infty}e^{-\alpha\lambda\omega}\le e^{-\alpha t}\frac{pe^{\alpha t_0}}{1-e^{-\alpha\omega}}\le\frac{p}{1-e^{-\alpha\omega}}.$$

The second sum is estimated in a similar way. Our result in the present paper is the following

Theorem 1. *Let conditions* **A1**–**A10** *hold. Then there exists a number* $\varepsilon_* \in (0, \varepsilon_0]$ *such that for* $\varepsilon \in (0, \varepsilon_*)$ *system* (1) *has a unique almost periodic solution* $x(t, \varepsilon)$ *depending continuously on* ε *and such that* $x(t, \varepsilon) \to \psi(t)$ *as* $\varepsilon \to 0$.

3 PROOF OF THE MAIN RESULT

In system (1) we change the variables according to the formula

$$x = \psi(t) + z$$

and obtain the system

$$\begin{aligned} \dot{z}(t) &= A(t)z(t) + (Qz)(t) + (R_\varepsilon z)(t), \quad t \neq t_k, \\ \Delta z(t_k) &= B_k z(t_k) + S_k z + \delta_{k\varepsilon} z, \quad k \in \mathbb{Z}, \end{aligned} \tag{7}$$

where

$$\begin{aligned} (Qz)_i(t) &= \sum_{j=1}^{m} b_{ij}(t) \left\{ f_j\left(\psi_j(t)+z_j(t)\right) - f_j(\psi_j(t)) - \frac{df_j}{dx_j}(\psi_j(t))z_j(t) \right\} \\ &\quad + \sum_{j=1}^{m} \left[g_{ij}\left(t, \psi_j(t-1) + z_j(t-1)\right) - g_{ij}(t, \psi_j(t-1) \right], \\ (S_k z)_i &= I_{ik}\left(\psi(t_k) + z(t_k), \psi(t_k - 1) + z(t_k - 1)\right) \\ &\quad - I_{ik}(\psi(t_k), \psi(t_k - 1)) - \sum_{j=1}^{m} \frac{\partial I_{ik}}{\partial x_j}(\psi(t_k), \psi(t_k - 1)) z_j(t_k) \end{aligned}$$

are nonlinearities inherent to the generating system (3) and therefore independent of the fluctuation of the delay $\varepsilon\varphi(t)$, while

$$\begin{aligned} (R_\varepsilon z)_i(t) &= \sum_{j=1}^{m} \left[g_{ij}(t, x_j(t - 1 - \varepsilon\varphi(t))) - g_{ij}(t, x_j(t - 1)) \right], \\ (\delta_{k\varepsilon} z)_i &= I_{ik}(x(t_k), x(t_k - 1 - \varepsilon\varphi(t_k))) - I_{ik}(x(t_k), x(t_k - 1)) \end{aligned}$$

are increments due to this fluctuation. For the sake of brevity we still write x instead of $\psi + z$ in $R_\varepsilon z$ and $\delta_{k\varepsilon} z$.

We can formally consider (7) as a nonhomogeneous system of the form (5). Since ω_1/ω is irrational, the nonhomogeneities are almost periodic if $z(t)$ is almost periodic. Then its unique almost periodic solution $z(t)$ must satisfy an equality of the form (6) which in this case is the operator equation

$$z = \mathcal{U}_\varepsilon z,$$

where

$$\begin{aligned} \mathcal{U}_\varepsilon z(t) &\equiv \int_{-\infty}^{\infty} \Phi(t)G(t-\tau)\Phi^{-1}(\tau)(Qz)(\tau)\, d\tau + \int_{-\infty}^{\infty} \Phi(t)G(t-\tau)\Phi^{-1}(\tau)(R_\varepsilon z)(\tau)\, d\tau \\ &\quad + \sum_{k \in \mathbb{Z}} \Phi(t)G(t - t_k)\Phi^{-1}(t_k) S_k z + \sum_{k \in \mathbb{Z}} \Phi(t)G(t - t_k)\Phi^{-1}(t_k)\delta_{k\varepsilon} z \\ &\equiv \mathcal{Q}z(t) + \mathcal{R}_\varepsilon z(t) + \mathcal{S}z(t) + \mathcal{D}_\varepsilon z(t). \end{aligned}$$

An almost periodic solution $x(t) = x(t, \varepsilon)$ of system (1) corresponds to a fixed point z of the operator $\mathcal{U}_\varepsilon$ in a suitable set of almost periodic functions. To this end we shall prove that $\mathcal{U}_\varepsilon$ maps a suitably chosen set into itself (§3.2) as a contraction (§3.3).

We first need to introduce some

3.1 Notation

For a vector $x \in \mathbb{R}^m$ we denote by $|\cdot|$ its sup $-$norm. We suppose that the spaces $AP_m\{t_k\}$ and ap_m respectively of almost periodic functions and almost periodic sequences are equipped with the norms

$$\|h\| = \sup_{t\in\mathbb{R}} |h(t)|, \qquad \|\{c_k\}_{k\in\mathbb{Z}}\| = \sup_{k\in\mathbb{Z}} |c_k|.$$

There exists a constant μ_0 such that the domain Ω contains a closed μ_0-neighbourhood Ω_1 of the periodic orbit $\{x = \psi(t) :\ t \in \mathbb{R}\}$. Let $\Omega_1^i,\ i = \overline{1,m}$, be the $i-$th projection of Ω_1. Let us denote

$$b = \sup\left\{|b_{ij}(t)| :\ t \in [0,\omega],\ i,j = \overline{1,m}\right\},$$

$$M_0 = \max\left\{ \sup\left\{ \left| -a_i(t)x_i + \sum_{j=1}^m b_{ij}(t)f_j(x_j) + \sum_{j=1}^m g_{ij}(t,y_j) + J_i(t) \right| :\ t \in [0,\omega],\ x,y \in \Omega_1,\ i = \overline{1,m} \right\},\right.$$
$$\left. \sup\left\{|I_{ik}(x,y)| :\ x,y \in \Omega_1,\ i = \overline{1,m},\ k = \overline{1,p}\right\} \right\},$$

$$M_1 = \max\left\{ \sup\left\{ \left| -a_i(t) + b_{ii}(t)\frac{df_i(x_i)}{dx_i} \right| :\ t \in [0,\omega],\ x_i \in \Omega_1^i,\ i = \overline{1,m} \right\},\right.$$
$$\sup\left\{ \left| b_{ij}(t)\frac{df_j(x_j)}{dx_j} \right| :\ t \in [0,\omega],\ x_j \in \Omega_1^j,\ i,j = \overline{1,m},\ j \neq i \right\},$$
$$\sup\left\{ \left| \frac{\partial g_{ij}}{\partial y_j}(t,y_j) \right| :\ t \in [0,\omega],\ y_j \in \Omega_1^j,\ i,j = \overline{1,m} \right\},$$
$$\sup\left\{ \left| \frac{\partial I_{ik}}{\partial x_j}(x,y) \right| :\ x,y \in \Omega_1,\ i,j = \overline{1,m},\ k = \overline{1,p} \right\},$$
$$\left. \sup\left\{ \left| \frac{\partial I_{ik}}{\partial y_j}(x,y) \right| :\ x,y \in \Omega_1,\ i,j = \overline{1,m},\ k = \overline{1,p} \right\} \right\};$$
$$\mathcal{M} = \sup\left\{ \|\Phi(t)\|\|\Phi^{-1}(\tau)\| :\ t,\tau \in [0,\omega] \right\}.$$

Let L be a common Lipschitz constant for the first derivatives of $f_j(\cdot)$ $(j = \overline{1,m})$, $g_{ij}(t,\cdot)$ $(t \in [0,\omega],\ i,j = \overline{1,m})$ and $I_{ik}(\cdot,\cdot)$ $(i = \overline{1,m},\ k = \overline{1,p})$ for $x,y \in \Omega_1$, whose existence is provided by conditions **A2**–**A4** and the compactness of the set Ω_1. For the sake of brevity we use the Landau symbol $O(\mu^r)$ for a quantity whose norm can be estimated by a constant times μ^r for μ small enough. The meaning of $O(\varepsilon)$ is similar.

For $a, b \in \mathbb{R}$ denote

$$]a,b[= \begin{cases} (a,b) & \text{if } a < b, \\ (b,a) & \text{if } a > b, \\ \emptyset & \text{if } a = b. \end{cases}$$

We may note that

$$\tau \in]t_k(\varepsilon), t_k + 1[\quad \Longleftrightarrow \quad t_k \in]\tau - 1, \tau - 1 - \varepsilon\varphi(\tau)[.$$

Define the "bad" set $\Delta_1^\varepsilon = \bigcup_{k\in\mathbb{Z}}]t_k(\varepsilon), t_k + 1[$. If $\varepsilon > 0$ is small enough, then Δ_1^ε is a disjoint union of intervals. We further define the "good" set $\Delta_2^\varepsilon = \mathbb{R} \setminus \Delta_1^\varepsilon$.

For the sake of convenience we assume that for $k = \overline{1,p}$ $t_k + 1 \neq t_\ell$ $\forall \ell \in \mathbb{Z}$. Then for $\varepsilon > 0$ small enough the "bad" set Δ_1^ε contains none of the points $t_k,\ k \in \mathbb{Z}$.

Let $\varepsilon_0 > 0$ be so small that all the above assumptions are valid for $\varepsilon \in (0, \varepsilon_0]$.

For $\mu \in (0, \mu_0]$ define a set of functions

$$\mathcal{T}_\mu = \{z \in AP_m :\ \|z\| \le \mu\}.$$

We shall find a relationship between ε and μ so that the operator $\mathcal{U}_\varepsilon$ maps the set $\mathcal{T}_\mu$ into itself as a contraction.

3.2 Invariance of the set $\mathcal{T}_\mu$ under the action of the operator $\mathcal{U}_\varepsilon$

Let $z \in \mathcal{T}_\mu$. We shall estimate $\|\mathcal{U}_\varepsilon z\|$ using the representation

$$\mathcal{U}_\varepsilon z(t) = \mathcal{Q}z(t) + \mathcal{R}_\varepsilon z(t) + \mathcal{S}z(t) + \mathcal{D}_\varepsilon z(t)$$

and system (1).

First we have

$$\begin{aligned}
&(S_k z)_i \\
&= \sum_{j=1}^m \int_0^1 \left(\frac{\partial I_{ik}}{\partial x_j}(\psi(t_k)+sz(t_k), \psi(t_k-1)+sz(t_k-1)) - \frac{\partial I_{ik}}{\partial x_j}(\psi(t_k), \psi(t_k-1)) \right) ds \cdot z_j(t_k) \\
&+ \sum_{j=1}^m \int_0^1 \left(\frac{\partial I_{ik}}{\partial y_j}(\psi(t_k)+sz(t_k), \psi(t_k-1)+sz(t_k-1)) - \frac{\partial I_{ik}}{\partial y_j}(\psi(t_k), \psi(t_k-1)) \right) ds \cdot z_j(t_k-1),
\end{aligned}$$

thus

$$\begin{aligned}
|(S_k z)_i| &\le \sum_{j=1}^m \int_0^1 Ls(|z(t_k)| + |z(t_k-1)|)\, ds \cdot |z_j(t_k)| \\
&+ \sum_{j=1}^m \int_0^1 Ls(|z(t_k)| + |z(t_k-1)|)\, ds \cdot |z_j(t_k-1)| \\
&\le mL(|z(t_k)| + |z(t_k-1)|)^2/2
\end{aligned}$$

and

$$|\mathcal{S}z(t)| \le \frac{4\mathcal{M}mLCp}{1 - e^{-\alpha\omega}}\mu^2 = O(\mu^2). \tag{8}$$

Similarly, for $\tau \neq t_k$, $\tau \neq t_k + 1$ we have

$$\begin{aligned}
(Qz)_i(\tau) &= \sum_{j=1}^m b_{ij}(\tau) \int_0^1 \left(\frac{df_j}{dx_j}(\psi_j(\tau) + sz_j(\tau)) - \frac{df_j}{dx_j}(\psi_j(\tau)) \right) ds \cdot z_j(\tau) \\
&+ \sum_{j=1}^m \int_0^1 \left(\frac{\partial g_{ij}}{\partial x_j}(\tau, \psi_j(\tau-1)+sz_j(\tau-1)) - \frac{\partial g_{ij}}{\partial x_j}(\tau, \psi_j(\tau-1)) \right) ds \cdot z_j(\tau-1),
\end{aligned}$$

thus

$$\begin{aligned}
|(Qz)_i(\tau)| &\le \sum_{j=1}^m \left(b \int_0^1 Ls|z_j(\tau)|\, ds \cdot |z_j(\tau)| + \int_0^1 Ls|z_j(\tau-1)|\, ds \cdot |z_j(\tau-1)| \right) \\
&\le mL\left(b|z(\tau)|^2 + |z(\tau-1)|^2\right)/2
\end{aligned}$$

and

$$|\mathcal{Q}z(t)| \le \frac{\mathcal{M}mL(b+1)C}{\alpha}\mu^2 = O(\mu^2). \tag{9}$$

Further on, let us denote for brevity $t_{ks\varepsilon} = t_k - 1 - s\varepsilon\varphi(t_k)$. Since the interval $]t_k(\varepsilon), t_k + 1[$ contains none of the points t_ℓ, we have

$$
\begin{aligned}
&(\delta_{k\varepsilon} z)_i = \int_0^1 \frac{\partial}{\partial s} I_{ik}(x(t_k), x(t_{ks\varepsilon}))\, ds \\
&= \sum_{j=1}^m \int_0^1 \frac{\partial I_{ik}}{\partial y_j}(x(t_k), x(t_{ks\varepsilon})) \frac{\partial}{\partial s} x_j(t_{ks\varepsilon})\, ds \\
&= -\varepsilon\varphi(t_k) \sum_{j=1}^m \int_0^1 \frac{\partial I_{ik}}{\partial y_j}(x(t_k), x(t_{ks\varepsilon})) \dot{x}_j(t_{ks\varepsilon})\, ds \\
&= -\varepsilon\varphi(t_k) \sum_{j=1}^m \int_0^1 \frac{\partial I_{ik}}{\partial y_j}(x(t_k), x(t_{ks\varepsilon})) \Bigg(-a_j(t_{ks\varepsilon}) x_j(t_{ks\varepsilon}) \\
&+ \sum_{\nu=1}^m b_{j\nu}(t_{ks\varepsilon}) f_\nu(x_\nu(t_{ks\varepsilon})) + \sum_{\nu=1}^m g_{j\nu}(t_{ks\varepsilon}, x_\nu(t_{ks\varepsilon} - 1 - \varepsilon\varphi(t_{ks\varepsilon}))) + J_i(t_{ks\varepsilon}) \Bigg)\, ds,
\end{aligned}
$$

thus $|(\delta_{k\varepsilon} z)_i| \le \varepsilon m M_0 M_1$ and

$$
|D_\varepsilon z(t)| \le \frac{2\mathcal{M} m M_0 M_1 C p}{1 - e^{-\alpha\omega}} \varepsilon = O(\varepsilon). \tag{10}
$$

Further on, if $\tau \in \Delta_2^\varepsilon \setminus \{t_k\}_{k\in\mathbb{Z}}$, then

$$
\begin{aligned}
&g_{ij}(\tau, x_j(\tau - 1 - \varepsilon\varphi(\tau))) - g_{ij}(\tau, x_j(\tau - 1)) \\
&= -\varepsilon\varphi(\tau) \int_0^1 \frac{\partial g_{ij}}{\partial y_j}(\tau, x_j(\tau_{s\varepsilon})) \Bigg(-a_j(\tau_{s\varepsilon}) x_j(\tau_{s\varepsilon}) + \sum_{\nu=1}^m b_{j\nu}(\tau_{s\varepsilon}) f_\nu(x_\nu(\tau_{s\varepsilon})) \\
&+ \sum_{\nu=1}^m g_{j\nu}(\tau_{s\varepsilon}, x_\nu(\tau_{s\varepsilon} - 1 - \varepsilon\varphi(\tau_{s\varepsilon}))) + J_i(\tau_{s\varepsilon}) \Bigg)\, ds,
\end{aligned}
$$

where $\tau_{s\varepsilon} = \tau - 1 - s\varepsilon\varphi(\tau)$, thus

$$
|g_{ij}(\tau, x_j(\tau - 1 - \varepsilon\varphi(\tau))) - g_{ij}(\tau, x_j(\tau - 1))| \le \varepsilon M_0 M_1.
$$

Let $\tau \in]t_k(\varepsilon), t_k + 1[$ for some $k \in \mathbb{Z}$. This means that the interval $]\tau - 1, \tau - 1 - \varepsilon\varphi(\tau)[$ contains just one discontinuity point t_k. Now we derive the estimate

$$
|g_{ij}(\tau, x_j(\tau - 1 - \varepsilon\varphi(\tau))) - g_{ij}(\tau, x_j(\tau - 1))| \le (\varepsilon + 1) M_0 M_1.
$$

We have

$$
\begin{aligned}
(\mathcal{R}_\varepsilon z)_i(t) &= \int_{-\infty}^{\infty} \Phi(t) G(t - \tau) \Phi^{-1}(\tau) (R_\varepsilon z)_i(\tau)\, d\tau \\
&= \int_{\Delta_1^\varepsilon} \Phi(t) G(t - \tau) \Phi^{-1}(\tau) (R_\varepsilon z)_i(\tau)\, d\tau \\
&+ \int_{\Delta_1^\varepsilon} \Phi(t) G(t - \tau) \Phi^{-1}(\tau) (R_\varepsilon z)_i(\tau)\, d\tau
\end{aligned}
$$

and

$$\begin{aligned}|(\mathcal{R}_\varepsilon z)_i(t)| &\le C\mathcal{M}\sum_{j=1}^{m}\left\{\int_{\Delta_1^\varepsilon} e^{-\alpha|t-\tau|}|g_{ij}(\tau,x_j(\tau-1-\varepsilon\varphi(\tau)))-g_{ij}(\tau,x_j(\tau-1))|\,d\tau\right.\\ &\quad\left.+\int_{\Delta_2^\varepsilon} e^{-\alpha|t-\tau|}|g_{ij}(\tau,x_j(\tau-1-\varepsilon\varphi(\tau)))-g_{ij}(\tau,x_j(\tau-1))|\,d\tau\right\}\\ &\le C\mathcal{M}m\left\{\int_{\Delta_1^\varepsilon} e^{-\alpha|t-\tau|}(\varepsilon+1)M_0M_1\,d\tau+\int_{\Delta_2^\varepsilon} e^{-\alpha|t-\tau|}\varepsilon M_0M_1\,d\tau\right\}\\ &= C\mathcal{M}mM_0M_1\left\{\varepsilon\int_{-\infty}^{\infty} e^{-\alpha|t-\tau|}\,d\tau+\int_{\Delta_1^\varepsilon} e^{-\alpha|t-\tau|}\,d\tau\right\}\\ &= C\mathcal{M}mM_0M_1\left\{\frac{2\varepsilon}{\alpha}+\int_{\Delta_1^\varepsilon} e^{-\alpha|t-\tau|}\,d\tau\right\}.\end{aligned}$$

Now it remains to show that

$$\int_{\Delta_1^\varepsilon} e^{-\alpha|t-\tau|}\,d\tau=\sum_{k\in\mathbb{Z}}\int_{]t_k(\varepsilon),t_k+1[} e^{-\alpha|t-\tau|}\,d\tau=O(\varepsilon).$$

To this end we shall use the fact that

$$\operatorname{meas}\,]t_k(\varepsilon),t_k+1[\le\varepsilon \quad\text{and}\quad]t_k(\varepsilon),t_k+1[\subset[t_k+1-\varepsilon,t_k+1+\varepsilon]\quad\forall k\in\mathbb{Z}.$$

For $t\in\mathbb{R}$ we shall consider two possibilities:

a) t belongs to none of the segments $[t_k+1-\varepsilon,t_k+1+\varepsilon]$, $k\in\mathbb{Z}$. Then we may assume that $t_0+1+\varepsilon\le t\le t_1+1-\varepsilon$. Then for $k\in\mathbb{N}\cup\{0\}$ we have

$$\begin{aligned}\int_{]t_{-k}(\varepsilon),t_{-k}+1[} e^{-\alpha|t-\tau|}\,d\tau&=\int_{]t_{-k}(\varepsilon),t_{-k}+1[} e^{-\alpha(t-\tau)}\,d\tau=e^{-\alpha t}\int_{]t_{-k}(\varepsilon),t_{-k}+1[} e^{\alpha\tau}\,d\tau\\ &\le e^{-\alpha(t_0+1+\varepsilon)}\varepsilon e^{\alpha(t_{-k}+1+\varepsilon)}=\varepsilon e^{-\alpha t_0}e^{\alpha t_{-k}}\end{aligned}$$

and as in §2 we obtain

$$\sum_{k=0}^{\infty}\int_{]t_{-k}(\varepsilon),t_{-k}+1[} e^{-\alpha|t-\tau|}\,d\tau\le\varepsilon e^{-\alpha t_0}\sum_{k=0}^{\infty}e^{\alpha t_{-k}}\le\frac{p\varepsilon}{1-e^{-\alpha\omega}}.$$

In a similar way we show that

$$\sum_{k=1}^{\infty}\int_{]t_k(\varepsilon),t_k+1[} e^{-\alpha|t-\tau|}\,d\tau\le\frac{p\varepsilon}{1-e^{-\alpha\omega}}.$$

Thus,

$$\int_{\Delta_1^\varepsilon} e^{-\alpha|t-\tau|}\,d\tau\le\frac{2p\varepsilon}{1-e^{-\alpha\omega}}=O(\varepsilon).$$

b) t belongs to one of these segments, say, $t_0+1-\varepsilon\le t\le t_0+1+\varepsilon$. Clearly,

$$\int_{]t_0(\varepsilon),t_0+1[} e^{-\alpha|t-\tau|}\,d\tau\le\varepsilon$$

and

$$\sum_{k\in\mathbb{Z}\setminus\{0\}}\int_{]t_{-k}(\varepsilon),t_{-k}+1[} e^{-\alpha|t-\tau|}\,d\tau\le\frac{2p\varepsilon}{1-e^{-\alpha\omega}},$$

thus again

$$\int_{\Delta_1^\varepsilon} e^{-\alpha|t-\tau|}\,d\tau = O(\varepsilon)$$

and

$$|\mathcal{R}_\varepsilon z(t)| = O(\varepsilon). \tag{11}$$

From (8), (9), (10) and (11) we obtain

$$|\mathcal{U}_\varepsilon z(t)| = O(\mu^2) + O(\varepsilon),$$

i.e.,

$$|\mathcal{U}_\varepsilon z(t)| \le K_1\mu^2 + K_2\varepsilon \tag{12}$$

for some positive constants K_1 and K_2.

To provide the validity of the inequality $|\mathcal{U}_\varepsilon z(t)| \le \mu$, we first choose

$$\tilde{\mu}_0 = \min\left\{\mu_0, \frac{1}{2K_1}\right\}.$$

Then for any $\mu \in (0, \tilde{\mu}_0]$ we have $K_1\mu^2 \le \mu/2$ and inequality (12) takes on the form

$$|\mathcal{U}_\varepsilon z(t)| \le \mu/2 + K_2\varepsilon.$$

If we choose

$$\tilde{\varepsilon}(\mu) = \min\left\{\varepsilon_0, \frac{\mu}{2K_2}\right\},$$

then for any $\varepsilon \in (0, \tilde{\varepsilon}(\mu)]$ we have $k_2\varepsilon \le \mu/2$ and thus $\|\mathcal{U}_\varepsilon z\| \le \mu$, *i.e.*, the operator $\mathcal{U}_\varepsilon$ maps the set $\mathcal{T}_\mu$ into itself for $\mu \in (0, \tilde{\mu}_0]$ and $\varepsilon \in (0, \tilde{\varepsilon}(\mu)]$.

3.3 Contraction property of the operator $\mathcal{U}_\varepsilon$

Let $z', z'' \in \mathcal{T}_\mu$. Then

$$\begin{aligned}\mathcal{U}_\varepsilon z'(t) - \mathcal{U}_\varepsilon z''(t) = (\mathcal{Q}z'(t) - \mathcal{Q}z''(t)) \ + \ (\mathcal{R}_\varepsilon z'(t) - \mathcal{R}_\varepsilon z''(t))\\ + (\mathcal{S}z'(t) - \mathcal{S}z''(t)) \ + \ (\mathcal{D}_\varepsilon z'(t) - \mathcal{D}_\varepsilon z''(t)).\end{aligned}$$

Here we shall consider only

$$\mathcal{S}z'(t) - \mathcal{S}z''(t) = \sum_{k\in\mathbb{Z}} \Phi(t)G(t-t_k)\Phi^{-1}(t_k)(S_kz' - S_kz'').$$

We have

$$\begin{aligned}
&(S_kz')_i - (S_kz'')i\\
&= \Big(I_{ik}(\psi(t_k)+z'(t_k), \psi(t_k-1)+z'(t_k-1)) - I_{ik}(\psi(t_k)+z''(t_k), \psi(t_k-1)+z''(t_k-1))\Big)\\
&\qquad - \sum_{j=1}^m \frac{\partial I_{ik}}{\partial x_j}(\psi(t_k), \psi(t_k-1))\Big(z_j'(t_k) - z_j''(t_k)\Big)\\
&= \sum_{j=1}^m \int_0^1 \left[\frac{\partial I_{ik}}{\partial x_j}(\psi(t_k)+sz'(t_k)+(1-s)z''(t_k), \psi(t_k-1)+sz'(t_k-1)+(1-s)z''(t_k-1))\right.\\
&\qquad \left. - \frac{\partial I_{ik}}{\partial x_j}(\psi(t_k), \psi(t_k-1))\right] ds \cdot \Big(z_j'(t_k) - z_j''(t_k)\Big)\\
&+ \sum_{j=1}^m \int_0^1 \left[\frac{\partial I_{ik}}{\partial y_j}(\psi(t_k)+sz'(t_k)+(1-s)z''(t_k), \psi(t_k-1)+sz'(t_k-1)+(1-s)z''(t_k-1))\right.\\
&\qquad \left. - \frac{\partial I_{ik}}{\partial y_j}(\psi(t_k), \psi(t_k-1))\right] ds \cdot \Big(z_j'(t_k-1) - z_j''(t_k-1)\Big),
\end{aligned}$$

thus

$$\begin{aligned}
&|(S_k z')_i - (S_k z'')i| \\
&\le \sum_{j=1}^{m} \int_0^1 L\big[s\big(|z'(t_k)| + |z'(t_k-1)|\big) + (1-s)\big(|z''(t_k)| + |z''(t_k-1)|\big)\big]\, ds \\
&\quad \times \Big(|z'(t_k) - z''(t_k)| + |z'(t_k-1) - z''(t_k-1)|\Big) \\
&\le 4mL\mu\|z' - z''\|
\end{aligned}$$

and

$$|\mathcal{S}z'(t) - \mathcal{S}z''(t)| \le \frac{8\mathcal{M}mLCp}{1 - e^{-\alpha\omega}}\mu\|z' - z''\| = O(\mu)\|z' - z''\|.$$

In a similar way, using some ideas of §3.2, we obtain

$$\begin{aligned}
&|\mathcal{Q}z'(t) - \mathcal{Q}z''(t)\| = O(\mu)\|z' - z''\|, \ \ \|\mathcal{R}_\varepsilon z'(t) - \mathcal{R}_\varepsilon z''(t)\| = O(\varepsilon)\|z' - z''\|, \\
&\quad \|\mathcal{D}_\varepsilon z' - \mathcal{D}_\varepsilon z''\| = O(\varepsilon)\|z' - z''\|
\end{aligned}$$

and, finally,

$$|\mathcal{U}_\varepsilon z'(t) - \mathcal{U}_\varepsilon z''(t)| = (O(\mu) + O(\varepsilon))\|z' - z''\|,$$

i.e.,

$$|\mathcal{U}_\varepsilon z'(t) - \mathcal{U}_\varepsilon z''(t)| \le (K_3\mu + K_4\varepsilon)\|z' - z''\|$$

for some positive constants K_3 and K_4.

Choose an arbitrary number $q \in (0,1)$ and denote $\mu_1 = \min\left\{\tilde{\mu}_0, \frac{q}{2K_3}\right\}$ and $\varepsilon_* = \min\left\{\tilde{\varepsilon}(\mu_1), \frac{q}{2K_4}\right\}$. Then for any $\mu \in (0, \mu_1]$ and $\varepsilon \in [0, \varepsilon_*]$ we have $\|\mathcal{U}_\varepsilon z' - \mathcal{U}_\varepsilon z''\| \le q\|z' - z''\|$, *i.e.*, the operator $\mathcal{U}_\varepsilon$ maps the set $\mathcal{T}_\mu$ into itself as a contraction.

Thus the operator $\mathcal{U}_\varepsilon$ has a unique fixed point in $\mathcal{T}_\mu$, which is an almost periodic solution $z(t,\varepsilon)$ of system (7). Since $z \equiv 0$ is the unique almost periodic solution of system (7) for $\varepsilon = 0$, then $z(t,0) = 0$. Now $x(t,\varepsilon) = \psi(t) + z(t,\varepsilon)$ is the unique almost periodic solution of system (1) and $\lim_{\varepsilon\to 0} x(t,\varepsilon) = \psi(t)$. This completes the proof of the theorem.

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